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New Representation of the Self-Duality and Exact Solutions for Yang–Mills Equations

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In this paper, we found a new representation for self-duality . In addition, exact solution class of the classical $SU(2)$ Yang–Mills field in four-dimensional Euclidean space and two exact solution classes for $SU(2)$ Yang–Mills when ρ is a complex analytic function are also obtained.

KEY WORDS: Self-dual $SU(2)$; Yang–Mills fields; Gauge theory.

PACS numbers: 11.15.-q Gauge field theories, 11.15.Kc Semiclassical theories in gauge fields, 12.10.-g, 12.15.-y Yang–Mills fields

1. INTRODUCTION

The self-dual Yang–Mills (a system of for Lie algebra-valued functions of C^4) play a central role in the field of integrable systems and also play a fundamental role in several other areas of mathematics and physics (Khater *et al.*, 1999a,b, 2002; Ablowitz *et al.*, 2003).

In addition, the self-dual Yang–Mills are of great importance in their own right and have found a remarkable number of applications in physics and mathematics as well. These arise in the context of gauge theory (Rajaraman, 1989), in classical general relativity (Mason and Newman, 1989; Witten, 1979), and can be used as a powerful tool in the analysis of 4-manifolds (Donaldson, 1983).

Nonabelian gauge theories first appeared in the seminal work of Yang and Mills (1954) as a nonabelian generalization of Maxwells. The fact that the Yang–Mills have a natural geometric interpretation was recognized early on in the history of gauge theory (Zakharov and Shabat, 1972; Ablowitz *et al.*, 1973).

The Yang–Mills are a set of coupled, second-order partial differential equations in four dimensions for the Lie algebra-valued gauge potential functions A_μ, and are extremely difficult to solve in general. The self-dual Yang–Mills describe

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a connection for a bundle over the Grassmannian of two-dimensional subspaces of the twistor space (Corrigan, 1979a,b).

A very important property of the theory of nonabelian gauge fields is that the action functional has local minima in the Euclidean domain with nonvanishing field strength $F_{\mu\nu}$ (Abolwitz and Clarkson, 1991). The corresponding field configurations, which are often called pseudoparticles, have the self-dual or anti-self-dual field strength, and fall into topologically inequivalent classes labelled by an integer n, the Pontryagin index. The existence of these nonlocal minima was first pointed out by Belavin *et al.* (1975) who also exhibited the solution of the self-duality with $n = 1$ for an $SU(2)$ gauge group. Solutions of the self-duality with an arbitrary number of pesudoparticles were discovered by Witten (1979) and ‘t Hooft (1979).

In this paper, we present a new representation and exact solutions for the self-duality.

The paper is organized as follows: This introduction followed by the new representation of the self-duality in Section 2. In Section 3 we, found an exact solution class of the classical $SU(2)$ Yang–Mills field . Moreover, two exact solution classes for self-dual $SU(2)$ gauge fields on Euclidean space when ρ is a complex analytic function are given in Section 4.

2. NEW REPRESENTATION OF THE SELF-DUALITY

The essential idea of Yang and Mills (1954) is to consider an analytic continuation of the gauge potential A_μ into complex space where x_1, x_2, x_3 and x_4 are complex. The self-duality $F_{\mu\nu} = {}^*F_{\mu\nu}$ are then valid also in complex space, in a region containing real space where the $x's$ are real. Now consider four new complex variables $y, \overline{y}, z$ and $\overline{z}$ defined by

$$\sqrt{2}y = x_1 + ix_2, \qquad \sqrt{2}\overline{y} = x_1 - ix_2,$$
$$\sqrt{2}z = x_3 - ix_4, \qquad \sqrt{2}\overline{z} = x_3 + ix_4. \tag{1}$$

It is simple to check that the self-duality $F_{\mu\nu} = {}^*F_{\mu\nu}$ reduces to

$$F_{yz} = 0, \quad F_{\overline{y}\overline{z}} = 0, \quad F_{y\overline{y}} + F_{z\overline{z}} = 0. \tag{2}$$

Equations (2) can be immediately integrated, since they are pure gauge, to give (Chau and Yamanaka, 1992, 1993; Ge *et al.*, 1994)

$$A_y = D^{-1}D_y, \quad A_z = D^{-1}D_z, \quad A_{\overline{y}} = \overline{D}^{-1}\overline{D}_{\overline{y}}, \quad A_{\overline{z}} = \overline{D}^{-1}\overline{D}_{\overline{z}}, \tag{3}$$

where D and $\overline{D}$ are arbitrary 2×2 complex matrix functions of $y, \overline{y}, z$ and $\overline{z}$ with determinant $= 1$ (for $SU(2)$ gauge group) and $D_y = \partial y D$, etc. For real gauge fields $A_\mu \doteq -A_\mu^+$ (the symbol $\doteq$ is used for valid only for real values of x_1, x_2, x_3

and x_4), we require

$$\overline{D} \doteq (D^{+})^{-1}. \tag{4}$$

Gauge transformations are the transformations

$$D \longrightarrow DU, \quad \overline{D} \longrightarrow \overline{D}U, \quad U^{+}U = I, \tag{5}$$

where U is a 2×2 complex matrix function of y, $\overline{y}$, z and $\overline{z}$ with determinant $= 1$. Under transformation (5), (4) remains invariant. We now define the hermitian matrix J as (Corrigan it et al., 1978, 1979, 1983)

$$J \equiv D\overline{D}^{-1} \doteq DD^{+}. \tag{6}$$

J has the very important property of being invariant under the gauge transformation (5). The only nonvanishing field strengths in terms of J becomes

$$F_{u\overline{v}} = -\overline{D}^{-1}(J^{-1}J_u)_{\overline{v}}\overline{D}, \tag{7}$$

$(u, v = y, z)$ and the remaining self-duality (2) takes the form:

$$(J^{-1}J_y)_{\overline{y}} + (J^{-1}J_z)_{\overline{z}} = 0. \tag{8}$$

The action density in terms of J (Corrigan and Hasslacher, 1979) is

$$\phi(J) \equiv -\frac{1}{2}Tr F_{\mu\nu}F_{\mu\nu} = -2Tr(F_{y\overline{y}}F_{z\overline{z}} + F_{y\overline{z}}F_{\overline{y}z}), \tag{9}$$

where

$$F_{\mu\nu} = \partial\mu A_\nu - \partial\nu A_\mu - [A_\mu, A_\nu]. \tag{10}$$

Our construction begins by explicit parametrization of the matrix J

$$J = \begin{pmatrix} \phi & \overline{\rho} \\ \rho & \frac{1+\rho\overline{\rho}}{\phi} \end{pmatrix}, \tag{11}$$

and for real gauge fields $A_\mu \doteq -A_\mu^{+}$, we require $\phi \doteq real$, $\overline{\rho} \doteq \rho^*$ ($\rho^* \equiv$ $complex\ conjugate\ of\ \rho$). The self-duality (8) take the form

$$\frac{1}{2}(1 + \rho\overline{\rho})\partial\mu\partial\mu ln\phi - \frac{1}{2}\overline{\rho}\partial\mu\partial\mu\rho + \frac{\rho}{\phi}[\phi_y\overline{\rho}_{\overline{y}} + \phi_z\overline{\rho}_{\overline{z}}] + \frac{\overline{\rho}}{\phi}[\phi_y\rho_{\overline{y}} + \phi_z\rho_{\overline{z}}]$$
$$-\rho_y\overline{\rho}_{\overline{y}} + \rho_z\overline{\rho}_{\overline{z}}] = 0, \tag{12}$$

$$\phi\partial\mu\partial\mu\rho - \rho\partial\mu\partial\mu\phi + 2(\phi_{\overline{y}}\rho_y + \phi_{\overline{z}}\rho_z - \phi_y\rho_{\overline{y}} - \phi_z\rho_{\overline{z}}) = 0, \tag{13}$$

$$\phi\partial\mu\partial\mu\overline{\rho} - \overline{\rho}\partial\mu\partial\mu\phi + 2(\phi_y\overline{\rho}_{\overline{y}} + \phi_z\overline{\rho}_{\overline{z}} - \phi_{\overline{y}}\overline{\rho}_y - \phi_{\overline{z}}\overline{\rho}_z) = 0, \tag{14}$$

where $\partial\mu\partial\mu = 2(\partial y\partial\overline{y} + \partial z\partial\overline{z})$.

The positive definite Hermitian matrix $J = DD^+$ can be factored into a product upper and lower (or vice versa) triangular matrices as follows

$$J = RR^+ = R^I R^{I+},$$

$$R = \begin{pmatrix} \sqrt{\phi} & 0 \\ \frac{\rho}{\sqrt{\phi}} & \frac{1}{\sqrt{\phi}} \end{pmatrix}, \quad R^I = \begin{pmatrix} \frac{1}{\sqrt{\phi^I}} & \frac{\rho^I}{\sqrt{\phi^I}} \\ 0 & \sqrt{\phi^I} \end{pmatrix}, \tag{15}$$

$$\phi \doteq real, \quad \overline{\rho} \doteq \rho^*, \quad \phi^I \doteq real, \quad \overline{\rho}^I \doteq \rho^{I*}. \tag{16}$$

It is evident from (15) that one can choose a gauge so that $D = R$ or $D = R^I$ and it is easy to check that in both gauges the self-duality (12)–(14) (in the case of $D = R^I$ all the $\phi, \rho, \overline{\rho}$ are replaced by $\phi^I, \rho^I, \overline{\rho}^I$).

From (15) we see that $R^{-1}R^I$ is a unitary matrix so that we can always make a gauge transformation from the R gauge to the R^I gauge.

Theorem 1. *If $(\phi, \rho, \overline{\rho})$ satisfy (12)–(14) then so do $(\phi^I, \rho^I, \overline{\rho}^I)$ defined by* (Prasad, 1980)

$$\phi^I = \frac{\phi}{1 + \rho\overline{\rho}}, \quad \rho^I = \frac{\overline{\rho}}{1 + \rho\overline{\rho}}, \quad \overline{\rho}^I = \frac{\rho}{1 + \rho\overline{\rho}}.$$

3. EXACT SOLUTION CLASS OF THE CLASSICAL SU(2)YANG–MILLS FIELD

To obtain an exact solution class of the classical $SU(2)$ Yang–Mills field in four-dimensional Euclidean space, consider the system.

$$\frac{1}{2}(1 + \rho\overline{\rho})\partial\mu\partial\mu ln\phi - \frac{1}{2}\overline{\rho}\partial\mu\partial\mu\rho + \frac{\rho}{\phi}[\phi_y\overline{\rho}_{\overline{y}} + \phi_z\overline{\rho}_{\overline{z}}] + \frac{\overline{\rho}}{\phi}[\phi_y\rho_{\overline{y}} + \phi_z\rho_{\overline{z}}]$$
$$-[\rho_y\overline{\rho}_{\overline{y}} + \rho_z\overline{\rho}_{\overline{z}}] = 0,$$
$$\phi\partial\mu\partial\mu\rho - \rho\partial\mu\partial\mu\phi + 2(\phi_{\overline{y}}\rho_y + \phi_{\overline{z}}\rho_z - \phi_y\rho_{\overline{y}} - \phi_z\rho_{\overline{z}}) = 0. \tag{17}$$

Let us make the ansatz (Kyriakopoulos, 1980)

$$\phi = \phi(g), \qquad \rho = e^{ia}\sigma(g). \tag{18}$$

Where $g = g(x_1, x_2, x_3, x_4)$ is a real function of x_μ, $\mu = 1, 2, 3, 4,$ ϕ and σ are real functions of g, and a is a real constant. Then (17) give, the relations

$$\left(g_{y\overline{y}} + g_{z\overline{z}}\frac{\phi^4}{2}\right)\left(\frac{1+\sigma^2}{\phi^2}\right)' + (g_y g_{\overline{y}} + g_z g_{\overline{z}})\phi^2\left[(1+\sigma^2)\frac{\phi'}{\phi} - \sigma\sigma'\right]' = 0, \tag{19}$$

$$(g_{y\overline{y}} + g_{z\overline{z}})\phi^2\left(\frac{\sigma}{\phi}\right)' + (g_y g_{\overline{y}} + g_z g_{\overline{z}})(\phi\sigma' - \sigma\phi')' = 0. \tag{20}$$

Where the prime means differentiation with respect to g. The above relations imply that the determinant of the coefficients of $(g_{y\overline{y}} + g_{z\overline{z}})$ and $(g_y g_{\overline{y}} + g_z g_{\overline{z}})$ is zero i.e.

$$\left(\frac{1+\sigma^2}{\phi^2}\right)'(\phi\sigma' - \sigma\phi')' - 2\left[(1+\sigma^2)\frac{\phi'}{\phi} - \sigma\sigma'\right]'\left(\frac{\sigma}{\phi}\right)' = 0. \qquad (21)$$

We shall determine ϕ and σ from the above (21), let $(\frac{1+\sigma^2}{\phi^2}) = u$, then $(\frac{1+\sigma^2}{\phi^2})' = u'$, (21) takes the form

$$(u'\phi^2)'\left(\frac{\phi\sigma' - \sigma\phi'}{\phi^2}\right) + u'(\phi\sigma' - \sigma\phi')' = 0, \qquad (22)$$

if we write $(u'\phi^2) = h$, $(\phi\sigma' - \sigma\phi') = w$, then (22) becomes

$$h'w + w'h = 0, \qquad (23)$$

from (23), we find $wh = c'$, then $w = c$, where c and c' are constants. Therefore we get finally

$$\phi = \sqrt{\frac{c}{2}}e^{-g}, \quad \sigma = \sqrt{\frac{c}{2}}e^{g},\ then\ \ \rho = \sqrt{\frac{c}{2}}e^{g+ia}. \qquad (24)$$

Applying theorem (1) to ϕ and ρ of (24), then we get

$$\phi^I = \frac{\sqrt{\frac{c}{2}}e^{-g}}{1+\frac{c}{2}e^{2g}},\ \rho^I = \frac{\sqrt{\frac{c}{2}}e^{g-ia}}{1+\frac{c}{2}e^{2g}},\ \overline{\rho}^I = \frac{\sqrt{\frac{c}{2}}e^{g+ia}}{1+\frac{c}{2}e^{2g}}. \qquad (25)$$

(24) and (25) is a new class of solutions of Yang–Mills for self-dual $SU(2)$ gauge fields.

4. EXACT SOLUTIONS FOR SELF-DUAL $SU(2)$ GAUGE FIELDS ON EUCLIDEAN SPACE WHEN ρ IS A COMPLEX ANALYTIC FUNCTION

Following (Khater *et al.*, 2004), we reduce the for self-dual $SU(2)$ gauge fields on Euclidean space to the following

$$\frac{1}{2}(1+\rho\overline{\rho})\partial\mu\partial\mu ln\phi - \frac{1}{2}\overline{\rho}\partial\mu\partial\mu\rho + \frac{\rho}{\phi}[\phi_y\overline{\rho}_{\overline{y}} + \phi_z\overline{\rho}_{\overline{z}}] + \frac{\overline{\rho}}{\phi}[\phi_y\rho_{\overline{y}} + \phi_z\rho_{\overline{z}}]$$
$$-[\rho_y\overline{\rho}_{\overline{y}} + \rho_z\overline{\rho}_{\overline{z}}] = 0,$$
$$\phi\partial\mu\partial\mu\rho - \rho\partial\mu\partial\mu\phi + 2(\phi_{\overline{y}}\rho_y + \phi_{\overline{z}}\rho_z - \phi_y\rho_{\overline{y}} - \phi_z\rho_{\overline{z}}) = 0. \qquad (26)$$

When ρ is a complex analytic function of y and z, then we have

$$\rho_{\overline{y}} = \rho_{\overline{z}} = 0, \quad \rho_{y\overline{y}} + \rho_{z\overline{z}} = 0. \qquad (27)$$

Then, the self-dual Yang–Mills (26) takes the form

$$\phi(\phi_{y\overline{y}} + \phi_{z\overline{z}}) - (\phi_y\phi_{\overline{y}} + \phi_z\phi_{\overline{z}}) = 0, \tag{28}$$

$$\rho(\phi_{y\overline{y}} + \phi_{z\overline{z}}) - (\rho_y\phi_{\overline{y}} + \rho_z\phi_{\overline{z}}) = 0. \tag{29}$$

We consider now two cases: (a) Let $\rho = \rho(\phi)$, then we find

$$\rho_y = \rho'\phi_y, \quad \rho_z = \rho'\phi_z. \tag{30}$$

Then the two Equations (28) and (29) becomes

$$\phi(\phi_{y\overline{y}} + \phi_{z\overline{z}}) - (\phi_y\phi_{\overline{y}} + \phi_z\phi_{\overline{z}}) = 0, \tag{31}$$

$$\rho(\phi_{y\overline{y}} + \phi_{z\overline{z}}) - \rho'(\phi_y\phi_{\overline{y}} + \phi_z\phi_{\overline{z}}) = 0. \tag{32}$$

If we do not consider the case $(\phi_{y\overline{y}} + \phi_{z\overline{z}}) = 0$ and $(\phi_y\phi_{\overline{y}} + \phi_z\phi_{\overline{z}}) = 0$, then we have

$$\phi\rho' - \rho = 0, \tag{33}$$

by integration we obtain

$$\rho = c\phi, \quad \text{where } c \text{ is complex constant.} \tag{34}$$

Both (31) and (32) reduce to the same. A solution is given by

$$\phi_y = \phi_z \qquad \phi_{\overline{y}} = \phi_{\overline{z}}. \tag{35}$$

The solution class is given by

$$\phi = F(y + z, \overline{y} - \overline{z}), \tag{36}$$

where F is an arbitrary function, (34) and (36) gives a new class of solutions of Yang–Mills for self-dual $SU(2)$ gauge fields. Applying theorem (1) to ϕ and ρ of (34) and (36), then we get

$$\phi^I = \frac{F}{1 + c\overline{c}F^2}, \quad \rho^I = \frac{\overline{c}F}{1 + c\overline{c}F^2}, \quad \overline{\rho}^I = \frac{cF}{1 + c\overline{c}F^2}. \tag{37}$$

(b) Let us make the ansatz

$$\phi = \phi(g), \qquad \rho = e^{ia}\sigma(g). \tag{38}$$

where $g = g(x_1, x_2, x_3, x_4)$ is a real function of x_μ, $\mu = 1, 2, 3, 4$, ϕ and σ are real functions of g, and a is a real constant. Then (28) and (29) give the relations

$$\phi\phi'(g_{y\overline{y}} + g_{z\overline{z}}) + (g_y g_{\overline{y}} + g_z g_{\overline{z}})\phi^2[\phi\phi'' - \phi'^2] = 0, \tag{39}$$

$$\sigma\phi'(g_{y\overline{y}} + g_{z\overline{z}}) + (g_y g_{\overline{y}} + g_z g_{\overline{z}})(\sigma\phi'' - \phi'\sigma') = 0. \tag{40}$$

where the prime means differentiation with respect to g. The above relations imply that the determinant of the coefficients of $(g_{y\overline{y}} + g_{z\overline{z}})$ and $(g_y g_{\overline{y}} + g_z g_{\overline{z}})$ is zero i.e.,

$$\frac{\sigma'}{\sigma} = \frac{\phi'}{\phi}, \tag{41}$$

by integrating(41), we obtain

$$\sigma(g) = c\phi(g), \quad \rho = ce^{ia}\phi(g). \tag{42}$$

Applying Theorem 1 to ϕ and ρ of (42), then we get

$$\phi^I = \frac{\phi(g)}{1 + c^2\phi^2(g)}, \quad \rho^I = \frac{ce^{-ia}\phi(g)}{1 + c^2\phi^2(g)}, \quad \overline{\rho}^I = \frac{ce^{ia}\phi(g)}{1 + c^2\phi^2(g)}. \tag{43}$$

(42) and (43) is a new class of solutions of Yang–Mills for self-dual $SU(2)$ gauge fields.

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